

# On Properties of the Sturm-Liouville Operator with Degenerate Boundary Conditions

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## Abstract

We consider spectral problems for the Sturm-Liouville operator with arbitrary complex-valued potential  $q(x)$  and degenerate boundary conditions. We solve corresponding inverse problem, and also study the completeness property and the basis property of the root function system.

**1. Introduction.** Consider the Sturm-Liouville equation

$$u'' - q(x)u + \lambda u = 0 \quad (1)$$

with two-point boundary conditions

$$B_i(u) = a_{i1}u'(0) + a_{i2}u'(\pi) + a_{i3}u(0) + a_{i4}u(\pi) = 0, \quad (2)$$

where the  $B_i(u)$  ( $i = 1, 2$ ) are linearly independent forms with arbitrary complex-valued coefficients and  $q(x)$  is an arbitrary complex-valued function of class  $L_1(0, \pi)$ . It is convenient to write conditions (2) in the matrix form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

and denote the matrix composed of the  $i$ th and  $j$ th columns of  $A$  ( $1 \leq i < j \leq 4$ ) by  $A(ij)$ ; we set  $A_{ij} = \det A(ij)$ .

It is known that conditions (2) can be divided into two classes:

- 1) nondegenerate conditions;
- 2) degenerate conditions.

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Boundary conditions (2) are called nondegenerate if they satisfy one of the following relations:

$$1) A_{12} \neq 0, \quad 2) A_{12} = 0, A_{14} + A_{23} \neq 0, \quad 3) A_{12} = 0, A_{14} + A_{23} = 0, A_{34} \neq 0.$$

There is an enormous literature related to the spectral theory of the Sturm-Liouville operator with nondegenerate boundary conditions. In particular, the following assertion has been proved.

**Theorem** ([1]). *For any nondegenerate conditions the spectrum of problem (1), (2) consists of a countable set  $\{\lambda_n\}$  of eigenvalues with only one limit point  $\infty$ , and the dimensions of the corresponding root subspaces are bounded by one constant. The system  $\{u_n(x)\}$  of eigen- and associated functions is complete and minimal in  $L_2(0, 1)$ ; hence, it has a biorthogonally dual system  $\{v_n(x)\}$ .*

In this paper, we study eigenvalue problems for the Sturm-Liouville operator with degenerate boundary conditions. This type of boundary conditions has been investigated much less.

## 2. Preliminaries.

Let boundary conditions (2) be degenerate. According to [1, 2], this is equivalent to the fulfillment of the following conditions:

$$A_{12} = 0, \quad A_{14} + A_{23} = 0, \quad A_{34} = 0.$$

According to [2], any boundary conditions of the considered class are equivalent to the boundary conditions determined by the matrix

$$A = \begin{pmatrix} 1 & b & 0 & 0 \\ 0 & 0 & 1 & -b \end{pmatrix}, \quad \text{or} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If in the first case  $b = 0$  then for any potential  $q(x)$  we have the initial value problem (the Cauchy problem) which has no eigenvalues. The same situation takes place in the second case.

Further we will consider the first case if  $b = \pm 1$ . Then the boundary conditions can be written in more visual form

$$u'(0) + (-1)^\theta u'(\pi) = 0, \quad u(0) + (-1)^{\theta+1} u(\pi) = 0. \quad (3)$$

$\theta = 0, 1$ . It is easily shown that if  $q(x) \equiv 0$  then any  $\lambda \in \mathbb{C}$  is an eigenvalue of infinite multiplicity. This abnormal example illustrates the difficulty of investigation of problems with boundary conditions of the considered class.

Denote by  $c(x, \mu), s(x, \mu)$  ( $\lambda = \mu^2$ ) the fundamental system of solutions to (1) with the initial conditions  $c(0, \mu) = s'(0, \mu) = 1$ ,  $c'(0, \mu) = s(0, \mu) = 0$ . The following identity is well known

$$c(x, \mu)s'(x, \mu) - c'(x, \mu)s(x, \mu) = 1. \quad (4)$$

Simple computations show that the characteristic equation of problem (1), (3) can be reduced to the form  $\Delta(\mu) = 0$ , where

$$\Delta(\mu) = c(\pi, \mu) - s'(\pi, \mu). \quad (5)$$

By  $\Gamma(z, r)$  we denote the disk of radius  $r$  centered at a point  $z$ . By  $PW_\sigma$  we denote the class of entire functions  $f(z)$  of exponential type  $\leq \sigma$  such that  $\|f(z)\|_{L_2(R)} < \infty$ , and by  $PW_\sigma^-$  we denote the set of odd functions in  $PW_\sigma$ .

### 3. Inverse problem.

The following two assertions provide necessary and sufficient conditions to be satisfied by the characteristic determinant  $\Delta(\mu)$ .

**Theorem 1.**[3] *If a function  $\Delta(\mu)$  is the characteristic determinant of a problem (1), (3), then*

$$\Delta(\mu) = \frac{f(\mu)}{\mu},$$

where  $f(\mu) \in PW_{\pi}^{-}$ .

**Theorem 2.** *Let a function  $v(\mu)$  have the form*

$$v(\mu) = \frac{f(\mu)}{\mu}, \quad (6)$$

where  $f(\mu) \in PW_{\pi}^{-}$ , and satisfies the condition

$$\int_{-\infty}^{\infty} |\mu^m f(\mu)|^2 d\mu < \infty, \quad (7)$$

where  $m$  is a nonnegative integer number. Then, there exists a function  $q(x) \in W_2^m(0, \pi)$  such that the characteristic determinant of problem (1), (3) with the potential  $q(x)$  satisfies  $\Delta(\mu) = v(\mu)$ .

**Proof.** If  $m = 0$  the theorem was proved in [3]. Further we will count that  $m > 0$ . Since [4]

$$|f(\mu)| \leq C_1 \|f(\mu)\|_{L_2(R)} e^{\pi |Im\mu|}, \quad (8)$$

it follows that there exists an arbitrary large positive integer  $N$  such that

$$|u(\mu)| < 1/10, \quad |f(\mu)| < 1 \quad (9)$$

on the set  $|Im\mu| \leq 1, Re\mu \geq N$ . Let  $\mu_n$  ( $n = 1, 2, \dots$ ) be a strictly monotone increasing sequence of positive numbers such that  $|\mu_n - (N + 1/2)| < 1/10$  if  $1 \leq n \leq N$  and  $\mu_n = n$  if  $n \geq N + 1$ . Consider the function

$$s(\mu) = \pi \prod_{n=1}^{\infty} \frac{\mu_n^2 - \mu^2}{n^2} = \frac{\sin \pi \mu}{\mu} \prod_{n=1}^N \frac{\mu_n^2 - \mu^2}{n^2 - \mu^2}. \quad (10)$$

Obviously, all zeros of the function  $s(\mu)$  are simple, and, in addition, the inequality

$$(-1)^n \dot{s}(\mu_n) > 0 \quad (11)$$

holds for any  $n$ . Denote  $P(\lambda) = \prod_{n=1}^N (\mu_n^2 - \lambda)$ ,  $Q(\lambda) = \prod_{n=1}^N (n^2 - \lambda)$ . Evidently,

$$\frac{\lambda^m P(\lambda)}{Q(\lambda)} = \lambda^m + \sum_{j=1}^m \alpha_j \lambda^{m-j} + \frac{R(\lambda)}{Q(\lambda)}, \quad (12)$$

where  $R(\lambda)$  is a polynomial of degree  $N - 1$  and  $\alpha_j$  are some constants. It follows from (12) that

$$\prod_{n=1}^N \frac{\mu_n^2 - \mu^2}{n^2 - \mu^2} = 1 + \sum_{j=1}^l \alpha_j \mu^{-2j} + \frac{R(\mu^2)}{\mu^{2l} Q(\mu^2)}. \quad (13)$$

It follows from (10), (13) that

$$\dot{s}(n) = \frac{\pi(-1)^n}{n} \left( 1 + \sum_{j=1}^m \alpha_j n^{-2j} + O(n^{-2m-2}) \right). \quad (14)$$

Consider the equation

$$z^2 - u(\mu_n)z - 1 = 0. \quad (15)$$

It has the roots

$$c_n^{\pm} = \frac{u(\mu_n) \pm \sqrt{u^2(\mu_n) + 4}}{2}. \quad (16)$$

It follows from (9) that for any  $n$  all numbers  $c_n^+$  lie in the disk  $\Gamma(1, 1/2)$  and all numbers  $c_n^-$  lie in the disk  $\Gamma(-1, 1/2)$ . Let for even  $n$   $c_n = c_n^+$ , and for odd  $n$   $c_n = c_n^-$ . Then  $(-1)^n \operatorname{Re} c_n > 0$  for any  $n = 1, 2, \dots$ . This, together with (11) implies that  $\operatorname{Re} w_n > 0$  for any  $n$ , where

$$w_n = \frac{c_n}{\mu_n \dot{s}(\mu_n)}. \quad (17)$$

We set  $F(x, t) = F_0(x, t) + \hat{F}(x, t)$ , where

$$F_0(x, t) = \sum_{n=1}^N \left( \frac{2c_n}{\mu_n \dot{s}(\mu_n)} \sin \mu_n x \sin \mu_n t - \frac{2}{\pi} \sin nx \sin nt \right),$$

$$\hat{F}(x, t) = \sum_{n=N+1}^{\infty} \left( \frac{2c_n}{\mu_n \dot{s}(\mu_n)} \sin \mu_n x \sin \mu_n t - \frac{2}{\pi} \sin nx \sin nt \right). \quad (18)$$

One can readily see that  $F_0(x, t) \in C^\infty(R^2)$ . Consider the function  $\hat{F}(x, t)$ . If  $n \geq N + 1$ , then, by taking into account (8), (16) and the rule for choosing the roots of equation (15), we obtain

$$c_n = (-1)^n + \frac{f(n)}{2n} + (-1)^n \left( \sum_{j=1}^m \beta_j \frac{f^{2j}(n)}{n^{2j}} + O(1/n^{2m+2}) \right). \quad (19)$$

It follows from (8), (14), (18), and (19) that

$$\begin{aligned} \hat{F}(x, t) &= \sum_{n=N+1}^{\infty} \frac{2}{\pi} \left( \frac{1 + (-1)^n \frac{f(n)}{2n} + \sum_{j=1}^m \beta_j \frac{f^{2j}(n)}{n^{2j}} + O(1/n^{2m+2})}{1 + \sum_{j=1}^m \alpha_j n^{-2j} + O(n^{-2m-2})} - 1 \right) \sin nx \sin nt = \\ &= \sum_{n=N+1}^{\infty} \frac{2}{\pi} \left[ \left( 1 + (-1)^n \frac{f(n)}{2n} + \sum_{j=1}^m \beta_j \frac{f^{2j}(n)}{n^{2j}} + O(1/n^{2m+2}) \right) \times \right. \\ &\quad \left. \left( 1 + \sum_{j=1}^m \tilde{\alpha}_j n^{-2j} + O(n^{-2m-2}) \right) - 1 \right] \sin nx \sin nt = \\ &= \frac{2}{\pi} \sum_{n=N+1}^{\infty} (\gamma_n f(n) + \sum_{j=1}^m \tilde{\alpha}_j n^{-2j} + O(n^{-2m-2})) \sin nx \sin nt = \\ &= (\hat{G}(x - t) - \hat{G}(x + t))/2, \end{aligned}$$

where

$$\begin{aligned} \hat{G}(y) &= \frac{2}{\pi} \sum_{i=1}^3 G_i(y), \quad G_1(y) = \sum_{n=N+1}^{\infty} \gamma_n f(n) \cos ny, \\ G_2(y) &= \sum_{n=N+1}^{\infty} \sum_{j=1}^m \tilde{\alpha}_j n^{-2j} \cos ny = \sum_{j=1}^m \tilde{\alpha}_j \sum_{n=N+1}^{\infty} n^{-2j} \cos ny, \\ G_3(y) &= \sum_{n=N+1}^{\infty} \tilde{\gamma}_n n^{-2m-2} \cos ny \end{aligned}$$

where  $|\gamma_n| < C_2/n$ ,  $|\tilde{\gamma}_n| < C_2$ .

The relation

$$\sum_{n=1}^{\infty} |nf(n)|^{2m} = \frac{1}{2} \|\mu^m f(\mu)\|_{L_2(R)},$$

which follows from the Paley-Wiener theorem, together with the Parseval equality and (7), implies that  $G_1(y) \in W_2^{m+1}[0, 2\pi]$ . It is known [5] that for any  $j = 1, 2, \dots$  the series  $\sum_{n=N+1}^{\infty} n^{-2j} \cos ny$  are infinitely differentiable functions on the segment  $[0, 2\pi]$ . One can readily see that  $G_3(y) \in W_2^{m+1}[0, 2\pi]$ . Therefore, we obtain the representation

$$F(x, t) = F_0(x, t) + (\hat{G}(x - t) - \hat{G}(x + t))/2, \quad (20)$$

where the functions  $F_0(x, t)$  and  $\hat{G}(y)$  belong to the above-mentioned classes.

Now let us consider the Gelfand-Levitan equation

$$K(x, t) + F(x, t) + \int_0^x K(x, s)F(s, t)ds = 0 \quad (21)$$

and prove that it has a unique solution in the space  $L_2(0, x)$  for each  $x \in [0, \pi]$ . To this end, it suffices to show that the corresponding homogeneous equation has only the trivial solution.

Let  $f(t) \in L_2(0, x)$ . Consider the equation

$$f(t) + \int_0^x F(s, t)f(s)ds = 0.$$

Following [6], by multiplying the last equation by  $\bar{f}(t)$  and by integrating the resulting relation over the interval  $[0, x]$ , we obtain

$$\begin{aligned} \int_0^x |f(t)|^2 dt + \sum_{n=1}^{\infty} \frac{2c_n}{\mu_n \dot{s}(\mu_n)} \int_0^x \bar{f}(t) \sin \mu_n t dt \int_0^x f(s) \sin \mu_n s ds - \\ - \sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^x \bar{f}(t) \sin nt dt \int_0^x f(s) \sin ns ds = 0. \end{aligned}$$

This, together with the Parseval equality for the function system  $\{\sin nt\}_1^\infty$  on the interval  $[0, \pi]$  implies that

$$\sum_{n=1}^{\infty} w_n \left| \int_0^x f(t) \sin \mu_n t dt \right|^2 = 0,$$

where the  $w_n$  are the numbers given by (17). Since  $Re w_n > 0$ , we see that  $\int_0^x f(t) \sin \mu_n t dt = 0$  for any  $n = 1, 2, \dots$ . Since [7, 8] the system  $\{\sin \mu_n t\}_1^\infty$  is complete on the interval  $[0, \pi]$ , we have  $f(t) \equiv 0$  on  $[0, x]$ .

Let  $\hat{K}(x, t)$  be a solution of equation (21), and let  $\hat{q}(x) = 2 \frac{d}{dx} \hat{K}(x, x)$ ; then it follows [9] from (20) that  $\hat{q}(x) \in W_2^m(0, \pi)$ . By  $\hat{s}(x, \mu)$ ,  $\hat{c}(x, \mu)$  we denote the fundamental solution system of equation (1) with potential  $\hat{q}(x)$  and the initial conditions  $\hat{s}(0, \mu) = \hat{c}'(0, \mu) = 0$ ,  $\hat{c}(0, \mu) = \hat{s}'(0, \mu) = 1$ . By reproducing the corresponding considerations in [6], we obtain  $\hat{s}(\pi, \mu) \equiv s(\mu)$ , whence it follows that the numbers  $\mu_n^2$  form the spectrum of the Dirichlet problem for equation (1) with potential  $\hat{q}(x)$ , and  $\hat{c}(\pi, \mu_n) = c_n$ , which, together with identity (4), implies that  $\hat{s}'(\pi, \mu_n) = 1/c_n$ .

Let  $\hat{\Delta}(\mu)$  be the characteristic determinant of problem (1), (3) with potential  $\hat{q}(x)$ . Let us prove that  $\hat{\Delta}(\mu) \equiv v(\mu)$ . By theorem 1, the function  $\hat{\Delta}(\mu)$  admits the representation

$$\hat{\Delta}(\mu) = \frac{\hat{f}(\mu)}{\mu},$$

where  $\hat{f}(\mu) \in PW_\pi^-$ . By taking into account relation (4) and the fact that the numbers  $c_n$  are roots of equation (15), we have

$$\hat{\Delta}(\mu_n) = \hat{c}(\pi, \mu_n) - \hat{s}'(\pi, \mu_n) = c_n - c_n^{-1} = v(\mu_n).$$



It follows that the function

$$\Phi(\mu) = \frac{u(\mu) - \hat{\Delta}(\mu)}{s(\mu)} = \frac{f(\mu) - \hat{f}(\mu)}{\mu s(\mu)}$$

is an entire function on the complex plane. Since the function  $g(\mu) = f(\mu) - \hat{f}(\mu)$  belongs to  $PW_{\pi}^{-}$ , it follows from (8) that

$$|g(\mu)| \leq C_3 e^{\pi |Im \mu|}. \quad (22)$$

From (10), we find that if  $|Im \mu| \geq 1$ , then

$$|\mu s(\mu)| \geq C_4 e^{\pi |Im \mu|} \quad (23)$$

( $C_4 > 0$ ). If  $|Im \mu| \geq 1$ , then we obtain the estimate  $|\Phi(\mu)| \leq C_3/C_4$ .

By  $H$  we denote the union of the vertical segments  $\{z : |Rez| = n + 1/2, |Im z| \leq 1\}$ , where  $n = N + 1, N + 2, \dots$ . It follows from (10) that if  $\mu \in H$ , then  $|\mu s(\mu)| \geq C_5 > 0$ . The last inequality, together with (22), (23), and the maximum principle for the absolute value of an analytic function, implies that  $|\Phi(\mu)| \leq C_6$  in the strip  $|Im \mu| \leq 1$ . Consequently, the function  $\Phi(\mu)$  is bounded on the entire complex plane; therefore, by the Liouville theorem, it is a constant. It follows from the Paley-Wiener theorem and the Riemann lemma [1] that if  $|Im \mu| = 1$ , then  $\lim_{|\mu| \rightarrow \infty} g(\mu) = 0$ , whence, we obtain  $\Phi(\mu) \equiv 0$ .

#### 4. Completeness and the basis property.

Completeness of the root function system of problem (1), (3) was investigated in [10]. In particular, it was shown that if  $q(x) \in C^k[0, \pi]$  for some  $k \geq 0$ , and  $q^{(k)}(0) \neq (-1)q^{(k)}(\pi)$ , then the root function system is complete in  $L_2(0, \pi)$ . If there exists an  $\varepsilon > 0$  such that  $q(x) - q(\pi - x) = 0$  for almost all  $x \in [0, \varepsilon]$ , then the mentioned system is not complete in  $L_2(0, \pi)$ . It was established in [3], [11]

that there exist potentials  $q(x)$  such that the root function systems of corresponding problems (1), (3) are complete in  $L_2(0, \pi)$  and contain associated functions of arbitrary high order, i.e. the dimensions of root subspaces infinitely grow.

Since for a wide class of potentials  $q(x)$  the root function system of problem (1), (3) is complete in  $L_2(0, \pi)$  one can set a question whether the mentioned system forms a basis.

Let  $\lambda_n = \mu_n^2$  ( $\operatorname{Re} \mu_n \geq 0, n = 1, 2, \dots$ ) be the eigenvalues of problem (1), (3) numbered neglecting their multiplicities in nondecreasing order of absolute value. By  $m(\lambda_n)$  we denote the multiplicity of an eigenvalue  $\lambda_n$ . In addition, assume that the function  $q(x)$  is continuous on the interval  $(0, \pi)$ .

**Theorem 3.** *Suppose a subsequence of eigenvalues  $\lambda_{n_k}$  satisfies the following two conditions:*

1.  $|\operatorname{Im} \mu_{n_k}| < M$ ;
2.  $\lim_{k \rightarrow \infty} \frac{m(\lambda_{n_k})}{\ln |\lambda_{n_k}|} = 0$ ;

*Then the system of eigenfunctions and associated functions of problem (1), (3) is not a basis in  $L_2(0, \pi)$ .*

**Proof.** Let us calculate the Green function  $G(x, \xi, \mu)$  of operator (1), (3). By [9],  $G(x, \xi, \mu) = H(x, \xi, \mu)/\Delta(\mu)$ , where  $H(x, \xi, \mu) = \Phi(x, \xi, \mu)/2 + g(x, \xi)\Delta(\mu)$ , where

$$\begin{aligned} \Phi(x, \xi, \mu) = & s(x, \mu) \{ c'(\pi, \mu) [-c(\xi, \mu)s(\pi, \mu) - s(\xi, \mu)(-1 - c(\pi, \mu))] - \\ & - [1 - c(\pi, \mu)] [c(\xi, \mu)(-1 + s'(\pi, \mu)) - s(\xi, \mu)c'(\pi, \mu)] \} - \\ & - c(x, \mu) \{ [1 + s'(\pi, \mu)] [-c(\xi, \mu)s(\pi, \mu) - s(\xi, \mu)(-1 - c(\pi, \mu))] + \\ & + s(\pi, \mu) [c(\xi, \mu)(-1 + s'(\pi, \mu)) - s(\xi, \mu)c'(\pi, \mu)] \}, \end{aligned} \quad (24)$$

$g(x, \xi) = \pm(s(x, \mu)c(\xi, \mu) - c(x, \mu)s(\xi, \mu))/2$ , the sign " + " is used for  $x > \xi$ , and the sign " - " is used for  $x < \xi$ . Combining like terms

in (24) gives

$$\begin{aligned} \Phi(x, \xi, \mu) = & 2[s(x, \mu)c(\xi, \mu) - c(x, \mu)s(\xi, \mu)] - \\ & -[c(\pi, \mu) + s'(\pi, \mu)][s(x, \mu)c(\xi, \mu) + c(x, \mu)s(\xi, \mu)] + \\ & + 2[c'(\pi, \mu)s(x, \mu)s(\xi, \mu) + s(\pi, \mu)c(x, \mu)c(\xi, \mu)]. \end{aligned} \quad (25)$$

Let  $e(x, \mu)$  be the solution of equation (1) satisfying the initial conditions  $e(0, \mu) = 1$ ,  $e'(0, \mu) = i\mu$ , and let  $K(x, t)$ ,  $K^+(x, t) = K(x, t) + K(x, -t)$ , and  $K^-(x, t) = K(x, t) - K(x, -t)$  be the transformation kernels [1] realizing the representations

$$\begin{aligned} e(x, \mu) &= e^{i\mu x} + \int_{-x}^x K(x, t)e^{i\mu t} dt, \\ c(x, \mu) &= \cos \mu x + \int_0^x K^+(x, t) \cos \mu t dt, \\ s(x, \mu) &= \frac{\sin \mu x}{\mu} + \int_0^x K^-(x, t) \frac{\sin \mu t}{\mu} dt. \end{aligned} \quad (26)$$

It was shown in [12] that

$$c(\pi, \mu) = \cos \pi \mu + \frac{\pi}{2} \langle q \rangle \frac{\sin \pi \mu}{\mu} - \int_0^\pi \frac{\partial K^+(\pi, t)}{\partial t} \frac{\sin \mu t}{\mu} dt, \quad (27)$$

$$s'(\pi, \mu) = \cos \pi \mu + \frac{\pi}{2} \langle q \rangle \frac{\sin \pi \mu}{\mu} + \int_0^\pi \frac{\partial K^-(\pi, t)}{\partial x} \frac{\sin \mu t}{\mu} dt, \quad (28)$$

where  $\langle q \rangle = \frac{1}{\pi} \int_0^\pi q(x) dx$ . By differentiating the second of equalities (26) and taking into account [12] that  $K^+(\pi, \pi) = \frac{\pi}{2} \langle q \rangle$ , we obtain

$$c'(\pi, \mu) = -\mu \sin \pi \mu + \frac{\pi}{2} \langle q \rangle \cos \pi \mu + \int_0^\pi \frac{\partial K^+(\pi, t)}{\partial x} \cos \mu t dt. \quad (29)$$

By substituting the right-hand sides of (26-29) in (25), we get

$$\begin{aligned}\Phi(x, \xi, \mu) &= 2(\sin \mu x \cos \mu \xi - \cos \mu x \sin \mu \xi)/\mu - \\ &\quad - 2 \cos \pi \mu (\sin \mu x \cos \mu \xi + \cos \mu x \sin \mu \xi)/\mu + \\ &\quad + 2(-\sin \pi \mu \sin \mu x \sin \mu \xi + \sin \pi \mu \cos \mu x \cos \mu \xi)/\mu + o(\mu^{-1})e^{\pi|Im\mu|} = \\ &= 2[\sin \mu(x - \xi) + \sin \mu(\pi - (x + \xi))]/\mu + o(\mu^{-1})e^{\pi|Im\mu|}.\end{aligned}$$

Throughout the following we assume that  $|Im\mu| < M$ . Then the last equality implies that

$$G(x, \xi, \mu) = \frac{R(x, \xi, \mu)}{\Delta(\mu)} + g(x, \xi), \quad (30)$$

where

$$R(x, \xi, \mu) = [\sin \mu(x - \xi) + \sin \mu(\pi - (x + \xi))]/\mu + o(\mu^{-1}). \quad (31)$$

Let us study the function  $G(x, \xi, \mu)$  in the neighborhood of the eigenvalues  $\lambda_n$ . It follows from [3] that each root subspace contains one eigenfunction and possibly associated functions. Let  $\{\overset{h}{u}_n(x)\}$  ( $h = \overline{0, m(\lambda_n)}$ ) be an arbitrary canonical system of eigenfunctions and associated functions of problem (1), (3), and let  $\{\overset{h}{v}_n(x)\}$  be appropriately normalized canonical system of eigenfunctions and associated functions of the adjoint boundary value problem [13], i.e.  $\overset{0}{u}_n(x)$  and  $\overset{0}{v}_n(x)$  are eigenfunctions, and  $\overset{h}{u}_n(x)$  and  $\overset{h}{v}_n(x)$  ( $h \geq 1$ ) are associated functions of order  $h$ , where

$$(\overset{h}{u}_n(x), \overset{g}{v}_k(x))_{L_2(0, \pi)} = \delta_{n,k} \delta_{h, m(\lambda_n) - 1 - g}.$$

Further we consider only root subspaces corresponding the mentioned-above subsequence of the eigenvalues  $\lambda_{n_k}$ . Denote

$$\overset{0}{R}_{n_k}(x, \xi) = \overset{0}{u}_{n_k}(x) \overline{\overset{0}{v}_{n_k}(\xi)},$$

$$R_{n_k}^{m(\lambda_{n_k})-1}(x, \xi) = \sum_{p=0}^{m(\lambda_{n_k})-1} \overline{u_{n_k}^p(x) v_{n_k}^{m(\lambda_{n_k})-1-p}(\xi)}.$$

Since the function  $f(\mu)$  has a root of multiplicity  $m(\lambda_{n_k})$  at the point  $\mu_{n_k}$ , then

$$f(\mu) = \sum_{l=m(\lambda_{n_k})}^{\infty} c_l(\mu - \mu_{n_k})^l = (\mu - \mu_{n_k})^{m(\lambda_{n_k})} \sum_{l=0}^{\infty} c_{m(\lambda_{n_k})+l}(\mu - \mu_{n_k})^l. \quad (32)$$

Obviously,  $c_{m(\lambda_{n_k})} = f^{(m(\lambda_{n_k}))}(\mu_{n_k})/m(\lambda_{n_k})!$ . Relations (30) and (32), together with [13] imply the equality

$$\begin{aligned} R_{n_k}^0(x, \xi) &= \lim_{\mu \rightarrow \mu_{n_k}} (\mu^2 - \mu_{n_k}^2)^{m(\lambda_{n_k})} G(x, \xi, \mu) = \\ &= \frac{2^{m(\lambda_{n_k})} m(\lambda_{n_k})! \mu_{n_k}^{m(\lambda_{n_k})+1} R(x, \xi, \mu_{n_k})}{f^{(m(\lambda_{n_k}))}(\mu_{n_k})}. \end{aligned} \quad (33)$$

From the Bernstein inequality [4], we obtain

$$|f^{(m(\lambda_{n_k}))}(\mu_{n_k})| \leq C_1 \pi^{m(\lambda_{n_k})}. \quad (34)$$

It follows from (33) and (34) that

$$|R_{n_k}^0(x, \xi)| \geq C_2 (2/\pi)^{m(\lambda_{n_k})} m(\lambda_{n_k})! |\mu_{n_k}|^{m(\lambda_{n_k})+1} |R(x, \xi, \mu_{n_k})|,$$

where  $C_2 > 0$ , hence,

$$\begin{aligned} &||R_{n_k}^0(x, \xi)||_{L_2((0,\pi) \times (0,\pi))}^2 \geq \\ &\geq C_3 [(2/\pi)^{m(\lambda_{n_k})} m(\lambda_{n_k})!]^2 |\mu_{n_k}|^{2m(\lambda_{n_k})+2} ||R(x, \xi, \mu_{n_k})||_{L_2((0,\pi) \times (0,\pi))}^2, \end{aligned} \quad (35)$$

where  $C_3 > 0$ .

If  $\overset{0}{v}_{n_k}(\xi) \neq 0$ , then the function  $\overset{m(\lambda_{n_k})-1}{R}_{n_k}(x, \xi)$  is an associated function of order  $m(\lambda_{n_k}) - 1$ , corresponding to the eigenvalue  $\mu_{n_k}^2$  and the eigenfunction  $\overset{0}{u}_{n_k}(x)$ . It follows from [14] that

$$\begin{aligned} & \|\overset{0}{R}_{n_k}(x, \xi)\|_{L_2(\pi/3, \pi/2)}^2 \leq \\ & \leq [C_4 m(\lambda_{n_k}) |\mu_{n_k}|]^{2m(\lambda_{n_k})-2} \|\overset{m(\lambda_{n_k})-1}{R}_{n_k}(x, \xi)\|_{L_2(\pi/4, 3\pi/4)}^2, \end{aligned} \quad (36)$$

where  $C_4$  is a constant independent of  $\xi$ . If  $\overset{0}{v}_{n_k}(\xi) = 0$ , then the validity of (36) is obvious. It follows from (36) and [15] that

$$\|\overset{0}{R}_{n_k}(x, \xi)\|_{L_2(0, \pi)}^2 \leq [C_5 m(\lambda_{n_k}) |\mu_{n_k}|]^{2m(\lambda_{n_k})-2} \|\overset{m(\lambda_{n_k})-1}{R}_{n_k}(x, \xi)\|_{L_2(0, \pi)}^2. \quad (37)$$

By integrating inequality (37) with respect to  $\xi$ , we have

$$\begin{aligned} & \|\overset{0}{R}_{n_k}(x, \xi)\|_{L_2((0, \pi) \times (0, \pi))}^2 \leq \\ & \leq [C_6 m(\lambda_{n_k}) |\mu_{n_k}|]^{2m(\lambda_{n_k})-2} \|\overset{m(\lambda_{n_k})-1}{R}_{n_k}(x, \xi)\|_{L_2((0, \pi) \times (0, \pi))}^2. \end{aligned} \quad (38)$$

It follows from (31) that

$$\|R(x, \xi, \mu_{n_k})\|_{L_2((0, \pi) \times (0, \pi))}^2 \geq C_7 |\mu_{n_k}|^{-2}, \quad (39)$$

where  $C_7 > 0$ .

Relations (35), (38), (39), and the Stirling formula imply that

$$\|\overset{m(\lambda_{n_k})-1}{R}_{n_k}(x, \xi)\|_{L_2((0, \pi) \times (0, \pi))}^2 \geq \frac{(m(\lambda_{n_k})! |\mu_{n_k}|)^2}{(C_8^{m(\lambda_{n_k})} m(\lambda_{n_k}))^{2m(\lambda_{n_k})}} \geq \frac{|\mu_{n_k}|^2}{C_9^{m(\lambda_{n_k})}},$$

where  $C_8, C_9 > 0$ . By the conditions of the theorem, the right-hand side of the last inequality tends to infinity as  $k \rightarrow \infty$ . This, combined with a resonance type theorem [16] implies the validity of theorem 3.

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